Lattice Quantum Chromodynamics

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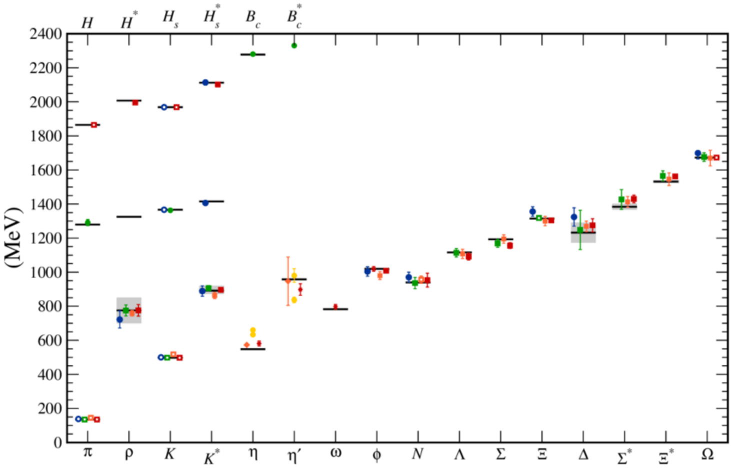
Overview

- Lattice QFT
  - Path integral formalism of QM/QFT
  - Euclidean vs. Minkowski path integral
  - Lattice: Discretized spacetime
  - Gauge invariance on the lattice: Wilson loop
  - Fermions in lattice QFT
- Numerical techniques: Monte Carlo
- Computational requirements

Hadron spectrum from lattice QCD



Black bars are the experimental values. Different color dots are various lattice QCD results. [Blum et al, 2013]

The Path Integral Formalism

- Time-slices: $i_i = t_0 < t_1 < \cdots < t_N = t_f$.
- Transition amplitude $\langle t_f, x_f \mid t_i, x_i \rangle$ is expressible as a path integral over all particle configurations:

$$\int e^{\mathrm{i}S[x]}\,\mathcal{D}x = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp\left(\mathrm{i}\int_{t_i}^{t_f} L(x(t), \dot{x}(t))\,\mathrm{d}t\right) \mathrm{d}x_0 \cdots \mathrm{d}x_N$$

- For fields we partition the spacetime manifold which induces a partition in the field configuration space. This gives us the path integral formulation for n-point functions:

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle = \frac{\int e^{\mathrm{i}S[\phi]} \phi(x_1) \cdots \phi(x_n)\,\mathcal{D}\phi}{\int e^{\mathrm{i}S[\phi]}\,\mathcal{D}\phi} = \text{see next page}$$

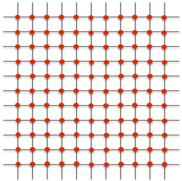
- This is the basic idea of lattice QFT!

Lattice: Discrete Partitioning of Finite Spacetime Volume

To put our QFT on a computer we must confine to a finite spacetime volume. This finite volume is partitioned by a finite number of lattice points. The path integral becomes:

$$\int \mathcal{D}\phi\, e^{\mathrm{i}S[x]} = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp\left(\mathrm{i} \int_V L(\phi(x^\mu))\mathrm{d}^4 x^\mu\right) \prod_i \mathrm{d}\phi(x_i^\mu)$$

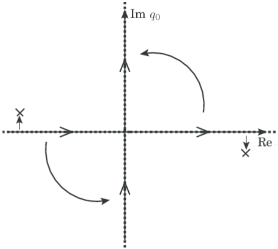
As the lattice spacing becomes zero this should approximate the continuous QFT value. This is how we define QFT on a lattice!



[From: `web.ma.utexas.edu`]

Euclidean Path Integral

- e^{iS} is highly oscillatory and therefore numerically unstable.
- To regularize it we Wick-rotate to Euclidean spacetime and compute e^{-S}.
- If we ever need to compare with Minkowski n-point function we simply Wick-rotate the Euclidean n-point function back to Minkowski spacetime at the end of the lattice calculation.



[From: Ori Yudilevich, Calculating Massive One-Loop Amplitudes in QCD]

Example: $SU(n)$ Gauge Theory

Criteria for a successful lattice implementation: (in increasing order of importance!)

- Regularity: the theory is completely finite since everything is regularized by the finite lattice spacing (UV) and finite lattice volume (infrared).
- Gauge invariance: we will see how gauge invariance is implemented in the Wilson theory in the next slide.
- Unitarity! Often taken for granted but is the most important.

Poincare symmetry is broken (obviously!) and is reduced to a finite subgroup: the symmetry group of the lattice. This includes the discrete translation symmetry and rotations in the four-dimensional hypercubic group.

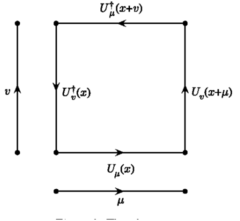
The usual C, P, T symmetries are still good!

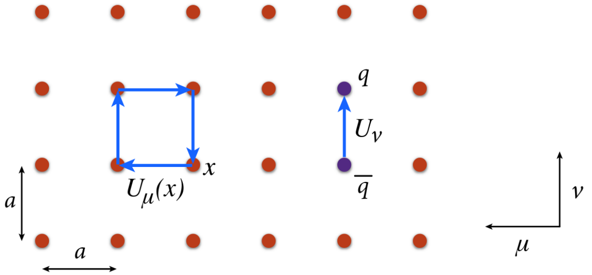
Gauge Theory: Fiber Bundles on a Manifold

- Gemoetric analogy with General Relativity.
- Connection on a principal $SU(n)$ bundle (a special fiber bundle with the Lie group $SU(n)$ as the fibers) is the gauge fields ${A_\mu}^i{}_j$. Here i and j are bundle indices.
- Curvature on the principal $SU(n)$ bundle gives us the force fields ${F_{\mu\nu}}^i{}_j$.
- In GR, curvature is defined as the change of a unit vector parallel transported along a unit loop.
- This analogy gives us the Wilson loop which implements gauge invariance in a lattice theory.

Wilson Loop

The simplest guess: going around a unit loop (a "plaquette") in the lattice space.





$$S_g = \frac{6}{g^2} \sum_{x,\mu,\nu} \left[1 - \frac{1}{3} \Re \left[\mathrm{tr}\, U_\mu(x) U_\nu(x + a\hat{\mu}) U^\dagger_\mu(x + a\hat{\nu}) U^\dagger_\nu(x) \right] \right]$$

Wilson Loop

In the continuous limit $a \to 0$, expanding in powers of a using

$$U(x) = \exp\left(iaA_\mu(x)\right)$$

and replacing the sum with an integral, one finds

$$S_g \to \int \mathrm{d}^4 x \, \frac{1}{4g^2} \operatorname{tr}\left[F_{\mu\nu}(x) F^{\mu\nu}(x)\right]$$

This is precisely the action for a $SU(n)$ guage theory! Here

$$F_{\mu\nu} = \partial_{[\mu} A_{\nu]} + i[A_\mu, A_\nu]$$

is the gauge force field. Note we have suppressed the bundle indices. Including bundle indices we have

$$\sum_{ij} {F_{\mu\nu}}^i{}_j {F^{\mu\nu j}}_i \quad \text{etc.}$$

Fermions in Lattice QFT

- The Fermion doubling problem: A naive Fermion action

$$D_\mu q(x) \to \frac{1}{2a}\left[U_\mu(x)q(x+a\hat{\mu}) - U_\mu(x-a\hat{\mu})^\dagger q(x-a\hat{\mu})\right]$$

  has two copies of fermions in the continuum limit.
- The Nielsen-Ninomiya theorem states that this is always the case. In d spacetime dimensions we get 2^d copies of fermions.
- A number of strategies to deal with this. See [Knechtli, Günther, Peardon 2017] for more details.

Numerical techniques: Monte Carlo

- The simplest Monte Carlo algorithm: To compute

$$I = \int_{\Omega} f(\vec{x}) \, \mathrm{vol}$$

where Ω is a region in $\mathbb{R}^m$, $\vec{x}$ a vector, and vol is the integral measure in $\mathbb{R}^m$ (Here m is very large!), we need the following

1. Generate N random sample points $\{\vec{x}_1, \dots, \vec{x}_N\}$ from a uniform distribution on Ω.
2. Compute

$$Q_N = \frac{V}{N} \sum_{i=1}^{N} f(\vec{x}_i)$$

3. In the limit $N \to \infty$ we have

$$\lim_{N \to \infty} Q_N = I$$

Computational Resources

- Lattice QCD is an extremely computationally intense program consisting primarily of matrix operations.
- These highly parallel computations are not suitable for CPUs which are more suited for branching operations (if-then-else statements).
- A relatively small scale program can be run on a GPU [R Babich 2011, B Joó 2012, 1612.07873, lattice.github.io/quda].
- Larger scale simulations require supercomputers [TOP500]. These are clusters of computer nodes connected by high-speed links capable of running large numbers of tasks in parallel.